Last Time: Row, Column, null spaces of matrix.

LINEAR (OPERATORS)*

NB: The textbook (Hefferon) calls these "Linear Transformations."

Defn: Let V be a vector space. A linear operator on V is a linear map $L: V \to V$.
i.e. a linear map w/ $\mathrm{dom}(L) = \mathrm{cod}(L)$.

Ex: $L: \mathbb{R}^3 \to \mathbb{R}^3$ w/ $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - 5y + z \\ x \\ 4x - 5y + z \end{pmatrix}$ *

Ex: The transpose is a linear operator on $M_{n,n}(\mathbb{R})$.
i.e. For square matrices.
↳ subEx: $T: M_{3\times3}(\mathbb{R}) \to M_{3\times3}(\mathbb{R})$ is an operator:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Note: The transpose (as an operator) is an automorphism; i.e. a self-isomorphism.

Ex: On $\mathcal{P}_n(\mathbb{R})$, $\frac{d}{dx}$ ← 1^{st} derivative operator is a linear operator! E.g. $n=3$:

$$\frac{d}{dx}\left[ax^3 + bx^2 + cx + d\right] = 3ax^2 + 2bx + c$$

is a linear operator: $\frac{d}{dx}[f + cg] = \frac{df}{dx} + c\frac{dg}{dx}$.

Ex (Generalization of previous example): Let

(*) $\mathcal{C}(\mathbb{R}) = \{ f : f$ has all derivatives, is a funct. on $\mathbb{R}\}$.

Then $\mathcal{C}(\mathbb{R})$ is a vector space w/ the usual scalar mult and vect. add. for functions.

Then $\boxed{\frac{d}{dx}}$ is a linear operator on $\mathcal{C}(\mathbb{R})$. ☺ ☒

Defn: Let V be a vector space. an automorphism of V is a linear isomorphism $L : V \to V$.

Ex: $L : \mathbb{R}^3 \to \mathbb{R}^3$ w/ $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - y \\ x + y + z \\ 2x - 2y - 5z \end{pmatrix}$

is a linear isomorphism, and therefore is an automorphism of $\mathbb{R}^3$.

very important...

Prop: Let V be a $\boxed{\text{finite dimensional}}$ V.S. and $L : V \to V$ be a linear operator. The following are equivalent.

① $\ker(L) = \{0_V\}$ (i.e. L is injective).

② $\operatorname{ran}(L) = V$ (i.e. L is surjective).

③ L is an automorphism.

Point: To decide if a linear operator is an automorphism, we need only check $\ker(L) = \{0_V\}$.

Ex: $\mathcal{P}_3(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_3(\mathbb{R})$ is NOT an automorphism...

B/C $\frac{d}{dx}[1] = 0$, but $1 \neq 0$. So, $1 \in \ker(\frac{d}{dx})$.

Ex: The transpose map $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is an automorphism. Indeed, If $M^T = 0_V$:

$$\begin{bmatrix} \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} \boxed{a} & \boxed{c} \\ \boxed{b} & \boxed{d} \end{bmatrix}$$

$\therefore \begin{cases} a = 0 \\ c = 0 \\ b = 0 \\ d = 0 \end{cases}$, so $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $\ker(T) = \{0_V\}$, and T is an automorphism ☒

Let's think about Linear Operators on $\mathbb{R}^n$.

In particular, suppose $L: \mathbb{R}^n \to \mathbb{R}^n$ is an automorphism.

Claim: L has an inverse map, L^{-1}.

i.e. There is a linear map $L^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ such that $L \circ L^{-1} = id_{\mathbb{R}^n} = L^{-1} \circ L$.

Recall: A linear map $L: \mathbb{R}^n \to \mathbb{R}^n$ has an associated matrix of transformation, $[L]_{\mathcal{E}_n}$:

i.e. the matrix $[L]_{\mathcal{E}_n}$ has columns the vectors $L(e_1), L(e_2), \cdots, L(e_n)$.

Ex: Consider $L: \mathbb{R}^3 \to \mathbb{R}^3$ w/ $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ 2y - 3z \\ x + 3y - 2z \end{pmatrix}$.

Then $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Note

$$L(e_1) = L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1 & +0 & +0\\ & 2\cdot 0 & -3\cdot 0\\ 1 & +3\cdot 0 & -2\cdot 0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix},$$

$$L(e_2) = L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \text{and} \quad L(e_3) = L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix},$$

so we have
$$[L]_{\mathcal{E}_n} = \begin{bmatrix}1 & 1 & 1\\ 0 & 2 & -3\\ 1 & 3 & -2\end{bmatrix} = \left[L(e_1) \,\vdots\, L(e_2) \,\vdots\, L(e_3)\right]$$

NB: This works because $L(\vec{x}) = [L]_{\mathcal{E}_n}\vec{x}$ ← C_i is the i^{th} column of $[L]_{\mathcal{E}_n}$

$$= \sum x_i \vec{C}_i$$ ← x_i is the i^{th} component of $\vec{x}$.

OTOH $L(\vec{x}) = L\left(\sum x_i e_i\right) = \sum x_i L(e_i)$. $\boxtimes$

Claim: Given L an automorphism of $\mathbb{R}^n$, we can compute L^{-1} via the following trick:

Observation 1: $\underline{\underline{L(\mathcal{E}_n)}}$ is a basis of $\mathbb{R}^n$.

Observation 2: If $L^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is the inverse of L, it must "undo" the transformation on $L(\mathcal{E}_n)$; i.e. $L^{-1}(L(e_i)) = e_i$

So this defines a map from basis $L(\mathcal{E}_n)$ to $\mathbb{R}^n$. By linearly extending this property, we obtain the inverse map $L^{-1}: \mathbb{R}^n \to \mathbb{R}^n$.

Ex: Consider the map $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix} & y & +z\\ x & & +z\\ x & +y & \end{pmatrix} = \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{row reduce to a matrix } [I_3 \mid M^{-1}].$$

I_3

$M = [L]_{\mathcal{E}_3}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{swap } \ell_1, \ell_2$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right] \quad \ell_3 - \ell_1 \to \ell_3$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \quad \ell_3 - \ell_2 \to \ell_3$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad -\tfrac{1}{2}\ell_3 \to \ell_3$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} \ell_1 - \ell_3 \to \ell_1 \\ \ell_2 - \ell_3 \to \ell_2 \end{array}$$

I_3 $\qquad M^{-1}$

$\therefore$ for $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ we see $M^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Claim: M^{-1} is the matrix of transformation for L^{-1}.

To verify this:

for $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we should have composition

$L^{-1} \circ L = id$ i.e. $M^{-1}\left(M\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e. $(M^{-1} \cdot M)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ i.e. $M^{-1} \cdot M = I_3$

Verify: $M^{-1}M = \frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} -1\cdot 0 + 1\cdot 1 + 1\cdot 1 & -1\cdot 1 + 1\cdot 0 + 1\cdot 1 & -1\cdot 1 + 1\cdot 1 + 1\cdot 0 \\ 1\cdot 0 - 1\cdot 1 + 1\cdot 1 & 1\cdot 1 - 1\cdot 0 + 1\cdot 1 & 1\cdot 1 - 1\cdot 1 + 1\cdot 0 \\ 1\cdot 0 + 1\cdot 1 - 1\cdot 1 & 1\cdot 1 + 1\cdot 0 - 1\cdot 1 & 1\cdot 1 + 1\cdot 1 - 1\cdot 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I_3 \quad \checkmark$$

Verify also $M \cdot M^{-1} = I_3$ (b/c we need $L \circ L^{-1} = id$).

↳ Exercise (check your matrix multiplication skills)...

$\boxtimes$

Point: Computing inverse transformations of automorphisms $L: \mathbb{R}^n \to \mathbb{R}^n$ can be done in 2 stages:

① Compute the matrix of the operator M.

② Row reduce $[M \mid I_n] \overset{*}{\rightsquigarrow} [I_n \mid M^{-1}]$

③ M^{-1} from step 2 is the matrix of transformation for $L^{-1}: \mathbb{R}^n \to \mathbb{R}^n$.

Remark: M^{-1} is the inverse matrix of M.

In particular, we defined (for an $n \times n$ matrix):

M^{-1} is the matrix of transformation of L_M^{-1}...